Socratic Mathematics
with Bill Carey

Session 4: Co-primality and Fractions

Outline:

Co-primality and Fractions

- In the last session we looked at a definition of co-primality and played with a rule to decide whether a pair of numbers are co-prime. One of the lovely things about mathematics is that it is full of inter-relatedness: ideas from one bit of mathematics show up in another part of mathematics, in the same way that, say, themes of love and economy connect Jane Austen's catalogue.
- The mathematical objects we'll play with this session are the bane of many a young student: fractions! You'll want to copy them onto the shared whiteboard for everyone to see together (I've written them the way I have on purpose, to illuminate some patterns; writing them in different ways will perhaps illuminate different patterns):

$$\frac{1}{2}$$

$$\frac{1}{3} \quad \frac{2}{3}$$

$$\frac{1}{4} \quad \frac{2}{4} \quad \frac{3}{4}$$

$$\frac{1}{5} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{4}{5}$$

$$\frac{1}{6} \quad \frac{2}{6} \quad \frac{3}{6} \quad \frac{4}{6} \quad \frac{5}{6}$$

$$\frac{1}{7} \quad \frac{2}{7} \quad \frac{3}{7} \quad \frac{4}{7} \quad \frac{5}{7} \quad \frac{6}{7}$$

- **Contemplation:** What do you notice about this list of fractions? What patterns do you see? What structure? There's a lot here! You could easily spend a handful of minutes in individual contemplation and another five or ten talking about what folks see in these fractions.
- **Definitions:** In our last session we introduced the idea of definitions, which act a bit like general rules we can apply to particular mathematical objects. From here on out, we'll have mathematical objects *and definitions* to play with.

They're good to share with the group and talk through a bit to make sure everyone's comfortable with them. Adding definitions allows us to engage with much more beautiful and interesting mathematics!

Two fractions are **equivalent** if they represent the same value. For example, $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent. (A good analogy here is baking: a half cup of flour is the same as two quarter-cups).

A fraction is **reduced** if it's written with the smallest denominator (bottom number) of all its equivalent fractions. For example, $\frac{1}{2}$ is reduced, but $\frac{2}{4}$ is not.

- **Discussion Questions:** Remember that the goal here is to seek out truth together, and convince yourselves that you've found it. As the facilitator, part of your responsibility is to make sure that everyone in the group is heard and on board!
  - In the row whose denominator (bottom number) is six, how many fractions are equivalent to another fraction in our big list?
  - Can you circle all the fractions in our big list that are equivalent to $\frac{1}{2}$?
  - Can you circle all the fractions in our big list that are reduced?
  - Of all the fractions with denominators less than twenty, which denominator has the most reduced fractions?
  - How is this list of fractions related to other things you've talked about in previous sessions?
- **Conclusion:** This relationship between co-primality and fractions was studied by one of the greatest mathematicians, Leonhard Euler, who wrote a description of these puzzles in 1775. In that note, he hoped for a pattern that would allow you to find out how many reduced fractions there are for any denominator easily. We are still looking for that pattern today.