CLASSICALU

Mathematics for Every Teacher

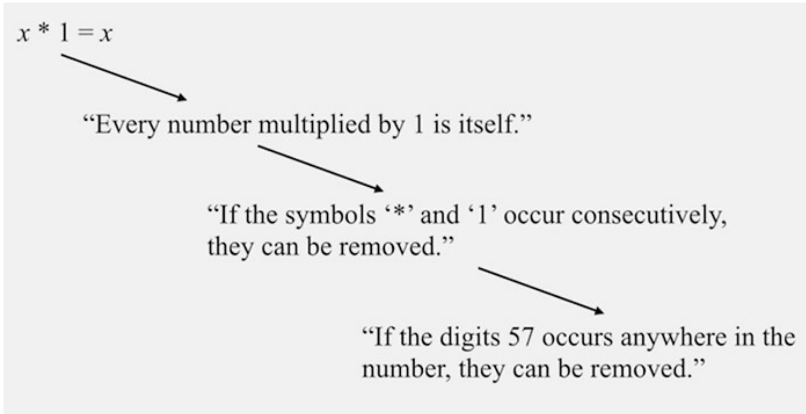
with Jake Tawney

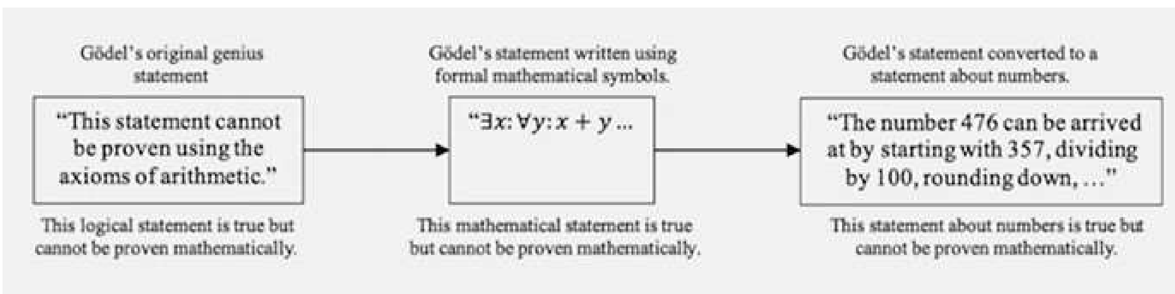
Lecture 12: Gödel's Incompleteness Theorems

Outline:

Gödel's Incompleteness Theorems, The Limited Power of Mathematical Proof

- The mathematical proof is so powerful because of its precision.
- Mathematicians insist on being precise in order to arrive at something that is true.
- **Russell's Paradox**: Sets are collections of objects. Some of those objects can also be sets themselves. It is even possible for a set to contain itself. Consider, then, the set of all sets that do not contain themselves. Does this set contain itself?
- Students have come to think of algebra as mere symbol manipulation rather than as symbols that reflect the reality of numbers.
- **Hilbert's Question:**
  - Is mathematics *complete*? Does every true mathematical statement in mathematics have a proof (even though we may not have discovered it yet)?
  - Is mathematics *consistent*? Can we ensure that no statement can be proved both true and false at the same time?
- **Gödel's First Incompleteness Theorem**: No formalized system of arithmetic can be both complete and consistent. In other words, so long as the system is not self-contradictory, there will always be true mathematical statements for which a proof is impossible.
  - "This statement cannot be proven using the axioms of arithmetic."
    - If this is false, then this statement can be proven using the axioms of arithmetic.
    - But if arithmetic is to be taken as consistence, then this is a logical contradiction, so this statement cannot be false.
    - However, suppose this statement is true. This statement is emphatically true, but it has no proof.
    - Math is incomplete.
    - He produced a statement that was true, but had no proof, and he translated that statement into an actual number. He found a way to have math talk about itself.
    - You can think of this as a mathematical process.
  - "Is there a proof for our mathematical statement?" can be turned into:
    - "Can we arrive at a particular number by applying certain mathematical operations?"
    - These are united by Gödel numbering.





- In proving that they are Gödel statements, then we have offered a proof that they are true. Remember that Gödel's theorem is about extremely formalized mathematics, not about mathematics generally.
- **Gödel's Second Incompleteness Theorem**: No formalized system of arithmetic can ever demonstrate that its axioms are consistent, that is that they do not lead to contradiction.
    - We can know that a statement is true. And even have a proof that is true, but this proof could never be arrived at by a formal system of mathematics (which could be automated by a computer). We can't arrive at the truth of that proof through an algorithm.
    - The human mind is not a computer. If it were purely algorithmic, we would not be able to discern the truth of those Gödel sentences. Our minds are not mere algorithm.
    - We can discern the truth of some things that even a computer cannot discern the truth.
    - "We must see the truth of a mathematical argument to be convinced of its validity...we reveal the very non-algorithmic nature of the seeing process itself.