Mathematics for Every Teacher

with Jake Tawney

Lecture 5: Existence of Irrational Numbers

Outline:

Are all numbers rational? Existence of Irrational Numbers

- Can all numbers be made from the four operations of addition, subtraction, division, and multiplication?
  - The reasonable answer is yes, but there is a fly in the ointment.
    - This is the right triangle. Can "c" be written as a ratio of a to b (a/b) where a and b are whole numbers?
    - Is "c" a rational number?
    - $c^2 = 2$, can c be written as the ratio of whole numbers?
    - Suppose that $c = \frac{m}{n}$ in lowest terms.
    - So $(\frac{m}{n})^2 = 2$
    - $\frac{m^2}{n^2} = 2$
    - $m^2 = 2n^2$
    - Claim: m^2 is even
    - Claim: m is even
    - $m = 2k$
    - $(2k)^2 = 2n^2$
    - $4k^2 = 2n^2$
    - $2k^2 = n^2$
    - Claim: n^2 is even, and so is n.
    - Claims: m is even and n is even?
      - But if m/n is in lowest terms, then they cannot both be even.
    - When we reach a contradiction, we prove something is false. When I reach nonsense I go back and look at the last time I made an assumption that was no proven, which was the original assertion that c could be written as the ratio of whole numbers. This is a proof by contradiction.
  - **Theorem:** There are numbers that exist that are not rational. That is, there are numbers that cannot be written as a fraction of integers. One such number is $\sqrt{2}$.
    - The only square roots that are rational are those built from square numbers.

- There is a claim that rational numbers all have decimals that repeat or terminate. The opposite claim is that irrational numbers neither repeat nor terminate.
    - The definition of irrational numbers is a number that cannot be written as a ratio of whole numbers.
    - It is true that their decimal expansions that does not repeat or terminate.
- Favorite two irrational numbers:
    - 0.1010010001…There is no repeating cycle of decimals, and yet this has a tremendous pattern.
    - 0.12345678910111213141516…This does not repeat or terminate, and yet the numbers have a pattern such that if you had a finite expansion you could determine the next digit.