Mathematics for Every Teacher

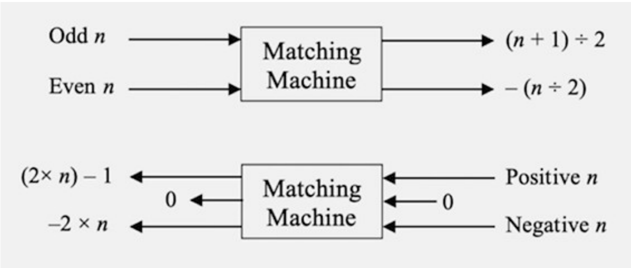
with Jake Tawney

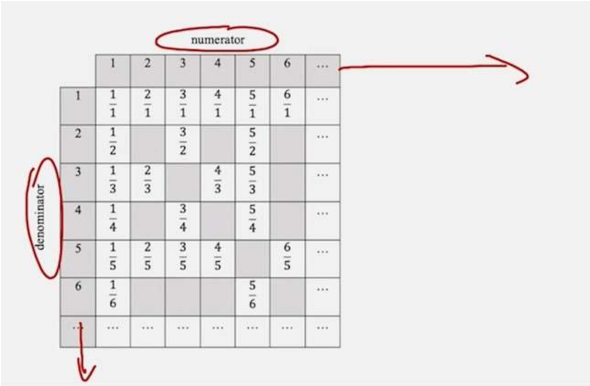
Lecture 6: How Many Irrational Numbers Are There?

Outline:

How many irrational numbers are there? How to measure infinity.

- Mathematics is the art of thinking deeply about simple things.
- How do we know how many of something there are?
- **Definition:** A collection has *n* things in it if there is an exact match between the objects in the collection and the numbers 1, 2, 3,…, *n.*
- **Definition:** Two finite collections have the same size if there is an exact match between the objects in the first set and the objects in the second set.
- What about the sizes of infinity?
    - How do we define sizes of infinity? Can you have a bigger infinity?
    - n always has to correspond with $n - 1$
    - $\infty + 1 = \infty$
    - $2\, x\, \infty = \infty + \infty = \infty$
    - $\infty\, x\, \infty = \infty^2$
        - Question: Does the size of integers equal the size of rational numbers?
            - The integers are the same size as the rational numbers.

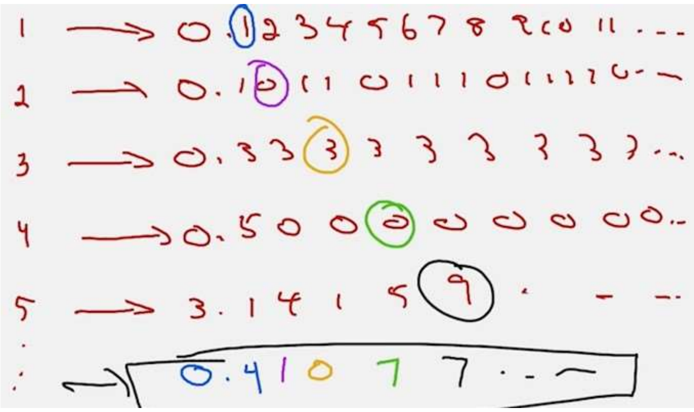




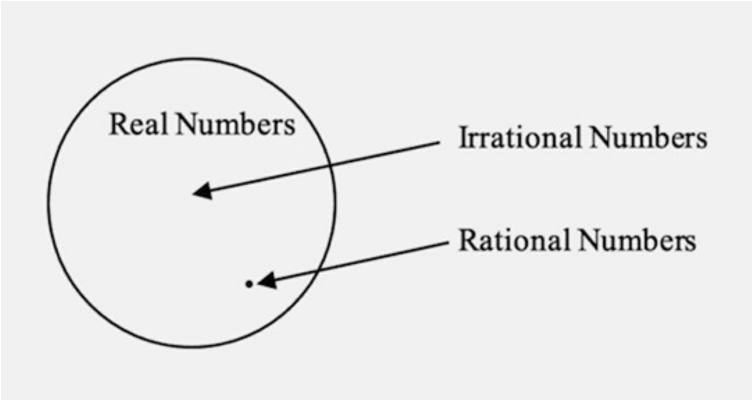
- Because of George Cantor’s traversing in a diagonal way of the matrix the matching machine works.

- We've defined the infinite sizes to be this one to one matching. Along the way every time we've been able to write down a matching from one set to the other in a way that I am convinced that nothing gets missed and that nothing gets repeated.

- **Equal Infinites Theorem:** The following collections of numbers all have the same size:
    - The Natural Numbers, ∞
    - The Integers, 2 *x* ∞
    - The Rational Numbers, ∞ *x* ∞
- The real numbers are much bigger though. This is a proof by contradiction:



- **The size of (some of) the Real Numbers Theorem:** There are more real numbers between 0 and 1 than there are integers. The real numbers between 0 and 1 are uncountable, or "not listable."



- **The Uncountability of the Irrational Numbers Theorem:** The irrational numbers are uncountable. Therefore, there are significantly more irrational numbers than rational numbers.
- Is there something above the real numbers and are there an infinite number of sizes of infinity?