ClassicalU

Teaching Math Classically
with Andrew Elizalde

Lesson 13: Mathematical Proofs Students Should Know

Outline:

Students can be engaged in proofs that require a demanding amount of logical reasoning.

- There may be a canon of "great proofs" they should be familiar with.
- They can learn to appreciate the aesthetic quality of the proof itself.

<u>Proof of the Infinitude of Primes</u>

- Whiteboard exercise *(time stamp 1:13)*
  - *Sieve of Eratosthenes:* the process of using the prime numbers chart in which non-prime numbers are crossed out, and prime numbers are circled (2, 3, 5, 7, 11, 13, 17, etc.)
  - *Prime numbers* are natural numbers that can only be divided by themselves and 1.
    - (1 is unique and can only be divided by itself and so it is ruled out.)
    - 2 is the first prime number. 3 is the second.
    - 4 is not because but is a *composite*. It has an additional factor beyond 1 and itself.
  - *(2:44)* Eratosthenes said we can work our way forward by considering multiples of 2, as they are not prime. Then 3, etc.
  - Mathematicians wondered how many prime numbers there are. Is there an end to them?
    - This is a great question to ask.
    - Mathematics has famous arguments to answer this question, and Euclid's answer to how many primes there are is one of them.
    - Students can make these arguments also.
    - This is Euclid's argument adapted.
      - Euclid started with a finite set of primes
      - Andrew is using a consecutive list of primes

On occasion, put on display beautiful mathematical arguments. This is a good occasion for a didactic mode for students to imitate, observe, and study.

- The argument assumes there is a finite number of primes and demonstrates it's not possible.
- *(6:17)* We begin with a set of primes.

$$P = \{P_1, P_2, P_3, P_4 \ldots P_K\}$$

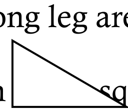
$$q = \{P_1, P_2, P_3, P_4 \ldots P_K\} + 1$$

$$q = \{2 * 3 * 5 * 7 \ldots P_K\} + 1$$

- *(7:40)* So q would have to be either a prime number bigger than P_k, or has a prime factor bigger than P_k.
    - q is either prime or composite. *(8:31)*
        - If q is prime, then P_k cannot be the last prime number.
        - If q is composite, then q can be broken down into $q = \# \text{ x } \#$. *(9:07)*
- *(9:28)* None of the prime numbers can be factors of q, since none of them divide by 2, 3, etc., because we added 1.
    - It is 1 more than a multiple of any of the prime numbers in our set.
- *(9:58)* q can't be in our list of primes.
    - *(10:10)* Therefore q is a number not in the list, or one of its factors is a number not in the list.
    - *(10:16)* That means that our original step, assuming that there is a finite number of primes, is not a fair move. Prime numbers cannot be contained.
    - There is an infinite number of primes.
- If students don't necessarily understand all the steps of this argument, at least it should be clear that the teacher took logical steps to get to that conclusion.
    - Students can do the same kind of logical reasoning.
    - Maybe students could memorize this compact argument.
- Part of the beauty of such an argument is that the idea is original and appealing.

<u>Pythagorean Theorem</u>

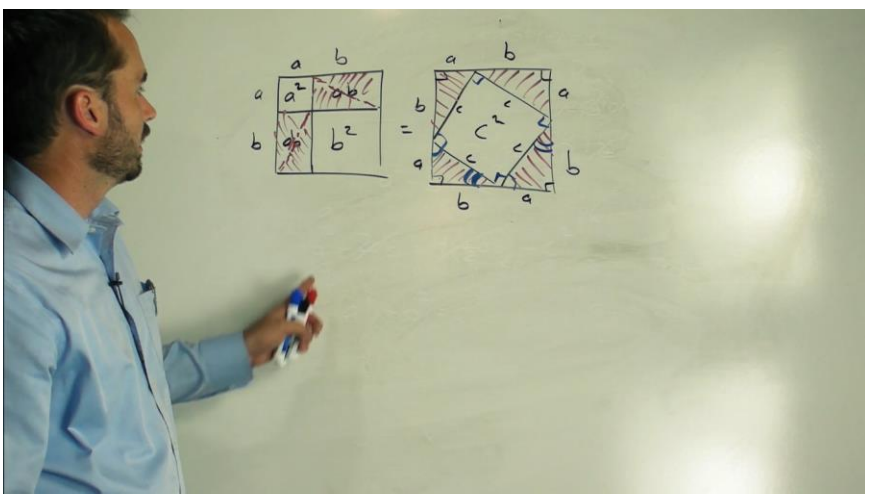
- *(12:50)* The shortest leg of a right triangle (a) and the longest leg (b) and the hypotenuse (c) are related in such a way that the square of the short leg plus the square of the long leg are equal to the square of the hypotenuse.



    - Or, the sum of the squares of the two sides of a right triangle is equal to the square of the hypotenuse.
    - $a^2 + b^2 = c^2$
- *What if we just made certain that before students use something, they understand where it came from?*
- There are literally hundreds of proofs for the Pythagorean Theorem.

- This one is Andrew's favorite among them because of its appeal.
- It is an ancient proof from India and China.
- It is likely older than Pythagoras himself.
- This one is not the most popular or common proof.

- Have students do this proof along with you. *(14:22–20:45)*
- Two squares of equal area are placed side by side. The first square is divided into a narrow rectangle and a broader rectangle, then divided again perpendicularly in the same fashion.
  - The sides are marked *ab* and *ab.*
  - The area of the square would then equal $a^2 + b^2 + 2ab$
- Divide the two *ab* squares into two equal triangles each and "transfer" those four triangles into the corners of the second square. The sides of the square in the middle are hypotenuses and could be labeled *c.*
  - The area of the second square could be written as $4(\frac{1}{2}ab) + c^2$
  - So then $a^2 + b^2 + 2ab = 4(\frac{1}{2}ab) + c^2$
  - $a^2 + b^2 = c^2$
- Since this proof is so accessible, have the students practice presenting it.

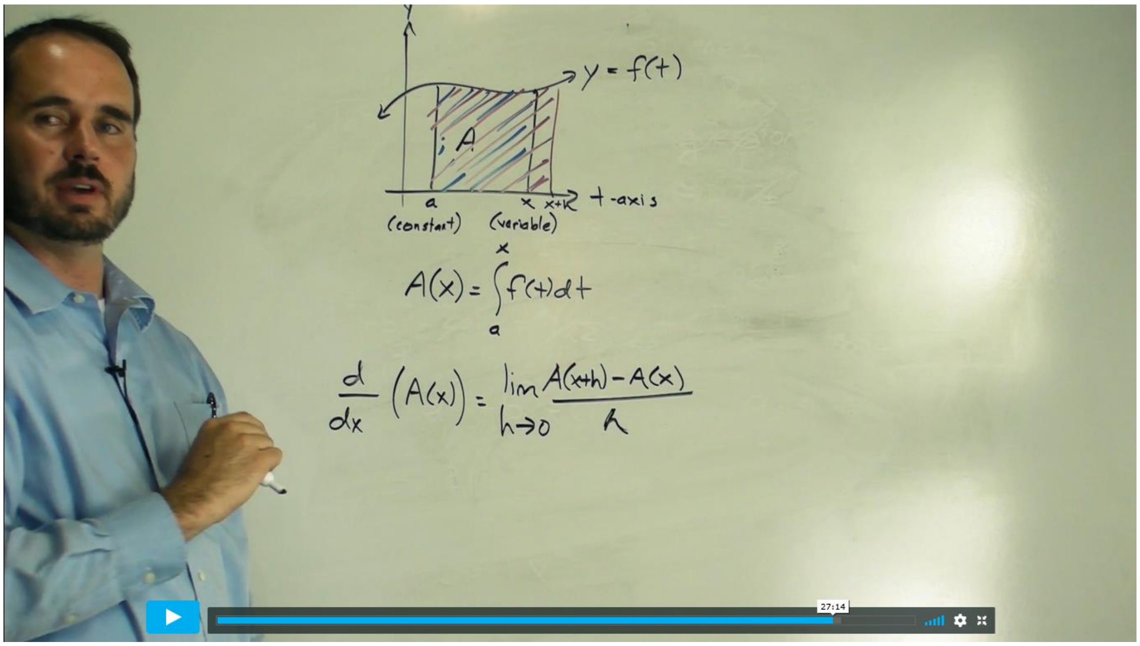


- The pictures alone can prove the argument without any statements.
  - In fact, certain ancient proofs do just that.
  - If you don't present this visual proof to your students, they might never be exposed to such a format.

<u>Andrew's Favorite Proof: **The Fundamental Theorem of Calculus**</u> *(23:45–29:50)*

- What are the aesthetic qualities of this proof?
  - algebra
  - logical reasoning/intuition

  - geometry
  - unexpectedness that makes it artistic
- The fundamental theorem of calculus asks this question: What happens when you take the derivative of this integral?



$$A(x) = \int_a^x f(t)dt$$

$$\frac{d}{dx}\big(A(x)\big) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h}$$

The area of the "rectangle" = $\lim_{h\to 0}\frac{h*f(x)}{h}$ ⟶ = f(x)

$$\frac{d}{dx}\big(A(x)\big) = f(x) \longrightarrow$$

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

- There exists a relationship between integration and differentiation.
  - One undoes the other.
  - Integration and differentiation are inversions of one another.
- Upon this argument we base our ability to evaluate definite integrals and establish corollaries.